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Endomorphism Monoids and Topological Subgraphs of Graphs

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We prove, that, *given a finite graph Y there exists a finite monoid* (semigroup with unity) *M such that any graph X whose endomorphism monoid is isomorphic to M contains a subdivision of Y*. This contrasts with several known results on the simultaneous prescribability of the endomorphism monoid and various graph theoretical properties of a graph. It is also related to the analogous problems on graphs having a given permutation group as a restriction of their automorphism group to an invariant subset.

1. INTRODUCTION

The *endomorphisms* of a graph X are those transformations to its vertex set $V(X)$ which send edges to edges. The endomorphisms of X form a monoid End X (under composition). The following is well known:

THEOREM 1.1. *Every monoid M is isomorphic to* End X *for some graph X.*

This holds with finite X for finite M [7–9]. (By graphs we mean undirected graphs, without loops and parallel edges.) Many results show that X can be chosen from particular classes of graphs: X may be required to have a given chromatic number $\geqslant 3$, to contain a given subgraph, to have prescribed connectivity, etc. [4, 5, 10, 11, 13]. It seems reasonable to seek naturally occurring and possibly not too restrictive graph theoretic properties which do not admit an arbitrary endomorphism monoid. We shall see that the non-containment of a subgraph, homeomorphic to a fixed finite graph, is such a property.

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The automorphism group Aut X of X consists of the invertible members of End X. An answer to the analogous question on automorphism groups was obtained in [2] as follows:

THEOREM 1.2. *Given a finite graph Y there is a finite group G such that any graph X whose automorphism group is isomorphic to G contains a subgraph contractible onto Y.*

The aim of the present note is to prove a stronger result under a stronger assumption, as formulated in the Abstract. This stronger result will fail if one is prescribing the automorphism group only (rather than the endomorphism monoid) in view of

THEOREM 1.3 (Frucht [6]). *Given a finite group G there exists a finite graph X such that* Aut $X \cong G$ *and every vertex of X has degree* 3.

PROBLEM 1.4. Let M be a finite semilattice with unity (i.e., a finite commutative idempotent monoid). Does there exist a graph X such that End $X \cong M$ and the maximum degree in X does not exceed a fixed finite bound, not depending on M?

What happens if we consider the class of all finite idempotent monoids M?

2. RESULTS AND PRELIMINARIES

For Y a graph, a subdivision of Y is obtained by *subdividing* the edges by new vertices of degree 2. The inverse of this operation will be called *kitting* edges of a graph (at vertices of degree 2). Both subdividing and kitting edges of Y result in graphs, *homeomorphic* to Y.

Let Ω be a set and Ω^Ω the monoid of all transformations of Ω. The submonoids of Ω^Ω are the *transformation monoids*, acting on Ω. A *permutation group* is a transformation monoid which is a group. If M is a monoid and $\varphi: M \to \Omega^\Omega$ is a homomorphism (sending the identity of M to the identity on Ω), then φ is called a *representation of M*. φ is *faithful* if it is injective.

Let M be a submonoid of Ω^Ω and V a set. $\varphi: M \to V^V$ is a *pseudorealization*, if it is a faithful representation, and there is an injective mapping of Ω into V ($x \mapsto \bar{x}$, say), such that

$$\bar{x}(\alpha\varphi) = \overline{x\alpha} \qquad (x \in \Omega, \quad \alpha \in M).$$

In other words, $M\varphi$ acts on a subset of V exactly in the way that M acts on Ω.

We shall say that the transformation monoid M' is a pseudorealization of the transformation monoid M if there is an isomorphism φ of M onto M'

which is a pseudorealization. Let us recall the following stronger version of Theorem 1.1:

THEOREM 2.1. *Given any transformation monoid M, there exists a graph X such that* End X *is a pseudorealization of* M [7, 12].

For the particular case of pseudorealizing a permutation group by Aut X, where X is a graph with prescribed graph theoretical properties, see [1, 3].

Our main result can now be formulated as follows:

THEOREM 2.1. *Let $\mathfrak{C}$ be a class of (not necessarily finite) graphs, closed under forming finite subgraphs and kitting edges (at vertices of degree 2). Assume that $\mathfrak{C}$ satisfies either:*

(i) *Given a finite monoid M there exists a graph $X \in \mathfrak{C}$ such that* End $X \cong M$; *or*

(ii) *given a finite permutation group P there exists a graph $X \in \mathfrak{C}$ such that* Aut X *is a pseudorealization of P.*

Then $\mathfrak{C}$ contains all finite graphs.

Note that by imposing condition (i) on $\mathfrak{C}$ we obtain the result formulated in the Abstract.

PROBLEM 2.3. Does the class of finite regular graphs satisfy (i)? (Note that it satisfies (ii) [3].)

3. THE PROOF OF THEOREM 2.2

First we prove that *if $\mathfrak{C}$ satisfies* (i) *then it satisfies* (ii) *as well.* This follows immediately from the following

LEMMA 3.1. *Given a permutation group P there exists a monoid M containing P such that*

(i) *P coincides with the group of invertible members of M;*

(ii) *every faithful representation of M is a pseudorealization.*

Proof. Let P act on the set Ω. For $|\Omega| = 2$ we can put $M = P$. Henceforth, let $|\Omega| \geqslant 3$.

Set

$$M = \{\alpha \in \Omega^{\Omega}\colon \alpha \in P \text{ or } |\Omega\alpha| \leqslant 2\}.$$

Obviously, M is a monoid and (i) holds. Denote by γ_x the constant mapping sending Ω to $x \in \Omega$. Let a and b denote two distinct elements of Ω.

Let $\varphi: M \to V^V$ be a faithful representation. Since $\gamma_a\varphi \neq \gamma_b\varphi$, there is a $v \in V$ with $v(\gamma_a\varphi) \neq v(\gamma_b\varphi)$. Put

$$\bar{x} = v(\gamma_x\varphi) \in V \qquad (x \in \Omega).$$

For $x \neq y$ we have $\bar{x} \neq \bar{y}$. Indeed, define $\alpha: \Omega \to \Omega$ by $x\alpha = a$, $z\alpha = b$ ($z \in \Omega - \{x\}$). Clearly, $\alpha \in M$, and

$$\begin{aligned}\bar{x}(\alpha\varphi) &= v(\gamma_x\varphi)(\alpha\varphi) = v((\gamma_x\alpha)\varphi) = v(\gamma_a\varphi)\\ &\neq v(\gamma_b\varphi) = v((\gamma_y\alpha)\varphi) = v(\gamma_y\varphi)(\alpha\varphi) = \bar{y}(\alpha\varphi),\end{aligned}$$

hence $\bar{x} \neq \bar{y}$.

Finally, for any $\beta \in M$ we have

$$\bar{x}(\beta\varphi) = v(\gamma_x\varphi)(\beta\varphi) = v((\gamma_x\beta)\,\varphi) = v(\gamma_{x\beta}\varphi) = \overline{x\beta}.$$

We conclude that φ is a pseudorealization, with $x \in \Omega$ corresponding to $\bar{x} \in V$. ▮

By Lemma 3.1, we have to prove only that Theorem 2.2 holds if $\mathfrak{C}$ satisfies (ii).

Let p be a prime number, Z_p the additive group of residue classes mod p, and $Z_p{}^2 = Z_p \oplus Z_p$. Set $\Omega = Z_p \times \{1, 2, 3\}$ and $\Omega_i = Z_p \times \{i\}$ ($i = 1, 2, 3$). So, $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3$ (disjoint union). Let us define the faithful representation $\varphi: Z_p{}^2 \to \Omega^\Omega$ by setting $\pi = (c, d)\varphi$ ($c, d \in Z_p$) with

$$\left.\begin{aligned}(a, 1)\pi &= (a + c, 1)\\ (a, 2)\pi &= (a + d, 2)\\ (a, 3)\pi &= (a + c + d, 3)\end{aligned}\right. \qquad \text{for every} \quad a \in Z_p.$$

Let $P = Z_p{}^2\varphi$. So, P is a permutation group of degree $3p$, isomorphic to $Z_p{}^2$. For $x \in \Omega$, $|P_x| = p$.

(For Q a permutation group, Q_x denotes the stabilizer of x.)

LEMMA 3.2. *Let p, Ω, and P be as above. Assume that X is a graph and* Aut X *is a pseudorealization of P. Then X contains a subdivision of $K_{p,p}$.*

($K_{p,p}$ denotes the complete bipartite graph with both color classes having p vertices.)

Proof. We may assume that Ω is a subset of $V(X)$, invariant under Aut X, and that P is the restriction of Aut X to Ω.

Set $A = \text{Aut } X$. By assumption, $A \cong Z_p{}^2$. Set $F_i = \{x \in V(X): |A_x| = p^iX\}$ ($i = 0, 1, 2$). Clearly, $\Omega \subseteq E_1$.

I. Assume that there is a path $a = x_0, \ldots, x_t = b$ such that $x_i \notin F_2$ $(i = 0, \ldots, t)$, a and b belong to F_1, and $A_a \neq A_b$. Let t be the minimum length of such paths $(t \geqslant 1)$. Then, clearly, $x_i \in F_0$ $(i = 1, \ldots, t-1)$. Let $\alpha \in A_a - \{id\}$, $\beta \in A_b - \{id\}$. Clearly, $A = \langle\alpha\rangle \oplus \langle\beta\rangle$. Set $a_i = a\beta^i$, $b_j = b\alpha^j$. The orbit of a under A is $\{a = a_0, \ldots, a_{p-1}\}$; the orbit of b is $\{b_0, \ldots, b_{p-1}\}$. These two sets are disjoint, since $A_a \neq A_b$ and A is Abelian. The path $(x_k\beta^i\alpha^j : k = 0, \ldots, t)$ connects a_i to b_j. We assert that these paths $(0 \leqslant i, j \leqslant p-1)$ form a subdivision of $K_{p,p}$. In order to see this, we only have to show that these p^2 paths are pairwise disjoint, apart from their endpoints. Assume that

$$x_k\beta^u\alpha^v = x_l\beta^r\alpha^s$$

for some integers k, l, u, v, r, s, $1 \leqslant k \leqslant t-1$ and $0 \leqslant l \leqslant t$. Setting $u - r = i$, $v - s = j$, we obtain

$$x_k\beta^i\alpha^j = x_l.$$

If $k < l$ then $x_0\beta^i\alpha^j, \ldots, x_k\beta^i\alpha^j = x_l, x_{l+1}, \ldots, x_t$ is a shorter walk satisfying our assumptions on $x_0, \ldots, x_t$, contradicting the minimality of t. $k > l$ is impossible for a similar reason, hence $k = l$. We conclude that $\beta^i\alpha^j \in A_{x_k}$. As $x_k \in F_0$ we infer that $\beta^i\alpha^j = id$, hence $i \equiv j \equiv 0 \bmod p$, proving the assertion.

II. Assume now that there is no path satisfying the assumptions of I. Let Ω_i' denote the union of those connected components of $x - F_2$ which meet Ω_i $(i = 1, 2, 3)$. Now, the sets Ω_i' are pairwise disjoint and they are invariant under Aut X.

Let α_i denote an automorphism of X acting nontrivially on Ω_i. Let β_i denote the permutation of $V(X)$ which agrees with α_i on Ω_i' and acts as the identity on $V(X) - \Omega_i'$. Clearly, $\beta_i \in$ Aut X, since β_i agrees with the automorphism α_i not only on Ω_i' but also on the neighbors of Ω_i' (namely, on $\Omega_i' \cup F_2$, F_2 being pointwise fixed under any automorphism).

The β_i's have order p and are pairwise disjoint, hence they generate a subgroup of order p^3 of Aut X, a contradiction. We conclude that case II is impossible, proving the Lemma. ∎

Now Theorem 2.2 follows: If Y is a finite graph, then, for some p, a subdivision of Y is contained in $K_{p,p}$. Hence, by Lemma 3.2, if Aut X is a pseudorealization of P for some $X \in \mathfrak{C}$, then $K_{p,p} \in \mathfrak{C}$ and so $Y \in \mathfrak{C}$. ∎

Remark 3.3. The permutation groups P in Lemma 3.2 are very particular. It would be desirable to find wide classes $\mathfrak{P}_n$ of finite permutation groups such that, if Aut X is a pseudorealization of any $P \in \mathfrak{P}_n$ then X contains a subdivision of K_n, the complete graph on n vertices.

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